

Irregularity of the Szegő Projection on Bounded Pseudoconvex Domains in \mathbb{C}^2

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Abstract. We construct bounded pseudoconvex domains in \mathbb{C}^2 for which the Szegő projection operators are unbounded on L^p spaces of the boundary for all $p \neq 2$.

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1. Introduction

Let Ω be a bounded domain with piecewise smooth boundary in \mathbb{C}^n . Let $d\sigma_E$ denote the Euclidean surface measure on the boundary $b\Omega$, the topological boundary of Ω , and let $L^p(b\Omega)$ denote the standard Lebesgue space with respect to $d\sigma_E$ on $b\Omega$. For $1 \leq p < \infty$, the Hardy space $H^p(b\Omega)$ is the closure of the set of holomorphic functions that are continuous up to the boundary in $L^p(b\Omega)$. When $p = 2$, there exists the orthogonal projection operator, so called the Szegő projection operator,

$$\mathbf{S}_\Omega : L^2(b\Omega, d\sigma_E) \rightarrow H^2(b\Omega)$$

where

$$\mathbf{S}_\Omega f(z_1, z_2) = \int_{b\Omega} \mathbb{S}_\Omega[(z_1, z_2), (t_1, t_2)] f(t_1, t_2) d\sigma_E(t_1, t_2).$$

The integral kernel, $\mathbb{S}_\Omega[(z_1, z_2), (t_1, t_2)]$, is called the Szegő kernel for $H^2(b\Omega)$. For details of the setup, we refer the reader to [Ste93] for domains with smooth enough boundary and to [Kno89] for piecewise smooth boundary like the bidisc.

It is a natural question to investigate the behavior of the integral operator \mathbf{S}_Ω on $L^p(b\Omega)$ for $p \neq 2$. It is known that under certain geometric conditions on the domain Ω the projection operator \mathbf{S}_Ω is bounded on

$L^p(b\Omega)$ for all $p \in (1, \infty)$, see for example [PS77, MS97, GP02, CD06, LS13]. On the other hand, \mathbf{S}_Ω does not have to be bounded on all $L^p(b\Omega)$ spaces. However, explicit examples of bounded pseudoconvex domains where \mathbf{S}_Ω is unbounded on $L^p(b\Omega)$ for all $p \neq 2$ is a new phenomena. On the other hand, in the complex plane \mathbb{C} , there are domains with piecewise smooth boundary whose Szegő projections are bounded for some (not for all $p \neq 2$) of $L^p(b\Omega)$ (cf. [Bék86, LS04]). In the Hardy space setting corresponding to the Shilov boundary, it has been known for a long time that the associated Szegő projection is unbounded for all $p \neq 2$ on irreducible bounded symmetric domains whose rank is greater than 1 (cf. [BB95]).

In this note, we present examples of domains for which the Szegő projection operators are L^p -irregular. More specifically, we construct bounded pseudoconvex domains in \mathbb{C}^2 (can be easily generalized to higher dimensions too) for which the Szegő projection operators are unbounded on L^p spaces for all $p \neq 2$. The construction follows the ideas in [Zey13] with modifications from Bergman kernels to Szegő kernels. Two main ingredients are the Forelli-Rudin inflation principle for Szegő kernels [Lig89, EZ06] and the L^p regularity of weighted Bergman projections on planar domains.

We work with the Euclidean surface measure $d\sigma_E$ on $b\Omega$ (the topological boundary¹) and hence with the Szegő projection operator associated to $d\sigma_E$. It is possible to put different measures on $b\Omega$ (or on the distinguished boundary), similar to Fefferman measure (see [Bar06]) or Leray measure (see [LS12, LS13]), and investigate the corresponding function theory. However, the domains constructed in this note are only piecewise smooth and contain weakly pseudoconvex points. Therefore, it is not clear how to define such measures that transform well under biholomorphic maps. We postpone the discussion of different measures and investigation of associated Hardy spaces and Szegő projections to a future paper.

2. Statement and Proof

Let \mathbb{D} be the unit disc in \mathbb{C} and

$$\phi(z) = \left(1 - |z|^2\right)^A \exp\left(\frac{-B}{\left(1 - |z|^2\right)^\alpha}\right)$$

for some $A \geq 0$, $B > 0$, and $\alpha > 0$. By using ϕ , we define the following complete Reinhardt domain in \mathbb{C}^2 ,

$$\Omega = \{(z_1, z_2) : z_1 \in \mathbb{D}, |z_2| < \phi(z_1)\}.$$

Note that for our choices of A, B and α the function $-\log \phi(z)$ is subharmonic on \mathbb{D} and consequently the corresponding domain Ω is pseudoconvex, see [Vla66, page 129].

¹Some authors work on the distinguished boundary of the domain, instead of the topological boundary.

Note also that a holomorphic function f on Ω is in $H^p(b\Omega)$ if

$$\sup_{U \subset \mathbb{D}} \sup_{\epsilon < \inf_U \phi} \int_{z_1 \in U, |z_2| = \phi(z_1) - \epsilon} |f|^p d\sigma_E < \infty. \quad (2.1)$$

As indicated in [Lig89, EZ06, Kno89], such functions have non-tangential boundary values almost everywhere on $b\Omega$ and

$$\|f\|_{H^p}^p = \int_{b\Omega} |f|^p d\sigma_E \quad (2.2)$$

where $d\sigma_E$ is explicitly written down by parametrizing the three real dimensional boundary and computing the Jacobian determinant (see [Lig89, EZ06])

$$\int_{b\Omega} \Phi d\sigma_E = \int_{\mathbb{D}} \int_0^{2\pi} \Phi(z_1, e^{i\theta} \phi(z_1)) \phi(z_1) \sqrt{1 + |\nabla \phi(z_1)|^2} d\theta dA(z_1).$$

The generalized Forelli-Rudin construction indicates the following representation (see [Lig89, EZ06]),

$$\mathbb{S}_\Omega[(z_1, z_2), (t_1, t_2)] = \sum_{j=0}^{\infty} z_2^j B_j(z_1, t_1) \overline{t_2^j} \quad (2.3)$$

where each $B_j(z_1, t_1)$ is the weighted Bergman kernel of the weight Bergman space

$$A^2\left(\mathbb{D}, c_j \phi^{2j+1} \sqrt{1 + |\nabla \phi|^2}\right)$$

(and \mathbf{B}_j labels the corresponding weighted Bergman projection) for some constant c_j . For the rest of the note, we set

$$\mu_j := c_j \phi^{2j+1} \sqrt{1 + |\nabla \phi|^2}.$$

Theorem 2.1. *Let Ω be as above. The Szegő projection operator \mathbf{S}_Ω , associated to the Euclidean surface measure, is bounded on $L^p(b\Omega)$ if and only if $p = 2$.*

Proof. It is clear that the operator is bounded when $p = 2$. We prove that it is unbounded for all $p \neq 2$. We use the representation (2.3) to show that if the weighted Bergman projection \mathbf{B}_0 is unbounded on $L^{p_0}(\mathbb{D}, \mu_0)$ for some $p_0 > 2$, then \mathbf{S}_Ω is also unbounded on $L^{p_0}(b\Omega)$.

In particular, let $f(z_1) \in L^{p_0}(\mathbb{D}, \mu_0)$ for some $p_0 > 2$ and let $F(z_1, z_2) = f(z_1)$ on Ω . Then,

$$\begin{aligned}
\|F\|_{L^{p_0}(b\Omega, d\sigma)}^{p_0} &= \int_{b\Omega} |F(z_1, z_2)|^{p_0} d\sigma_E \\
&= \int_{\mathbb{D}} \int_0^{2\pi} |F(z_1, e^{i\theta}\phi(z_1))|^{p_0} \phi(z_1) \sqrt{1 + |\nabla\phi(z_1)|^2} d\theta dA(z_1) \\
&= \int_{\mathbb{D}} \int_0^{2\pi} |f(z_1)|^{p_0} \phi(z_1) \sqrt{1 + |\nabla\phi(z_1)|^2} d\theta dA(z_1) \\
&= 2\pi \int_{\mathbb{D}} |f(z_1)|^{p_0} \phi(z_1) \sqrt{1 + |\nabla\phi(z_1)|^2} dA(z_1) \\
&= 2\pi \|f\|_{L^{p_0}(\mathbb{D}, \mu_0)}^{p_0}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathbf{S}_\Omega F(z_1, z_2) &= \int_{b\Omega} F(t_1, t_2) \left(\sum_{j=0}^{\infty} z_2^j B_j(z_1, t_1) \overline{t_2^j} \right) d\sigma_E(t_1, t_2) \\
&= \int_{\mathbb{D}} \int_0^{2\pi} F(t_1, t_2) \left(\sum_{j=0}^{\infty} z_2^j B_j(z_1, t_1) \overline{t_2^j} \right) d\theta \mu_0 dA(t_1) \\
&= 2\pi \int_{\mathbb{D}} f(t_1) B_0(z_1, t_1) \mu_0 dA(t_1) \\
&= 2\pi \mathbf{B}_0 f(z_1).
\end{aligned}$$

Therefore, it remains to show that the weighted Bergman projection operator \mathbf{B}_0 is unbounded on $L^{p_0}(\mathbb{D}, \mu_0)$. For this purpose, we invoke the following theorem [Zey13, Theorem 1.2] (see also [Dos04]) for the weighted Bergman projection \mathbf{B}_0 .

Theorem 2.2. *If λ is a radial weight on \mathbb{D} which satisfies*²

1. $\lambda(r)$ is a smooth function on $[0, 1]$,
2. $\frac{d^n}{dr^n} \lambda(1) = 0$ for all $n \in \mathbb{N}$,
3. for all $n \in \mathbb{N}$ there exists $a_n \in (0, 1)$ such that $(-1)^n \frac{d^n}{dr^n} \lambda(r)$ is non-negative on the interval $(a_n, 1)$.

Then the weighted Bergman projection \mathbf{B}_λ is bounded from $L^p(\mathbb{D}, \lambda)$ to $L^p(\mathbb{D}, \lambda)$ if and only if $p = 2$.

Recall that

$$\mu_0 = c_0 \phi \sqrt{1 + |\nabla\phi|^2}.$$

We have to check that μ_0 is a radially symmetric function and, as a function of $r = |z|$, it is smooth on $[0, 1]$. Moreover, $\frac{d^n}{dr^n} \mu_0(1) = 0$ for all $n \in \mathbb{N}$ and $(-1)^n \frac{d^n}{dr^n} \mu_0(r)$ is non-negative on $(a_n, 1)$ for some $0 < a_n < 1$ for all $n \in \mathbb{N}$.

²Here, we abuse the notation and consider λ as a function on $[0, 1]$ and by $\lambda(z)$ we mean $\lambda(|z|)$.

We go over this verification for the particular case $A = 0, B = 1$ and $\alpha = 1$; i.e. we take $\phi(z) = \exp\left(\frac{-1}{1-|z|^2}\right)$. Then a quick calculation gives (we drop c_0),

$$\mu_0(z) = \exp\left(\frac{-1}{1-|z|^2}\right) \sqrt{1 + \exp\left(\frac{-2}{1-|z|^2}\right) \frac{2|z|^2}{(1-|z|^2)^4}}.$$

It is clear that μ_0 is indeed a radial function and, as a function of $r = |z|$, it is smooth on $[0, 1]$. Furthermore we have

$$\mu_0(r) = \nu\left(\frac{1}{1-r^2}\right)$$

with

$$\nu(s) := e^{-s} \sqrt{1 - 2e^{-2s}(s^3 - s^4)}.$$

It then suffices to show that $\lim_{s \rightarrow \infty} \frac{d^n}{ds^n} \nu(s) = 0$ for all $n \in \mathbb{N}$ and $(-1)^n \frac{d^n}{ds^n} \nu(s)$ is non-negative on (s_n, ∞) for some $s_n > 0$ for all $n \in \mathbb{N}$. This easily follows from the following expression of $\frac{d^n}{ds^n} \nu(s)$,

$$\frac{d^n}{ds^n} \nu(s) = e^{-s} \frac{(-1)^n + \sum_{k=1}^n e^{-ks} P_k(s)}{(1 - 2e^{-2s}(s^3 - s^4))^{n-\frac{1}{2}}},$$

where P_k is a polynomial. This expression is proved by an induction argument. For the general case of A, B and α we get a similar pattern with an exponential term times some *lower order* terms.

This means μ_0 is a radial weight that decays exponentially on the boundary that satisfies all the other conditions in Theorem 2.2. We conclude that \mathbf{B}_0 is unbounded on $L^p(\mathbb{D}, \mu_0)$ for all $p \in (1, \infty)$, except $p = 2$. Therefore, the Szegő projection operator \mathbf{S}_Ω is also unbounded on $L^p(b\Omega)$ for all $p \in (1, \infty)$, except $p = 2$. \square

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